# Organizing Topic: Logarithmic Modeling

## Mathematical Goals:

- Students will model logarithmic relationships from data gathered during activities and from Internet database sources.
- Students will investigate and analyze key characteristics of logarithmic functions including domain, range, asymptotes, increasing/decreasing behavior, and end behavior.
- Students will use knowledge of transformations to write an equation given the graph of function and graph a function, given its equation.
- Students will make predictions using logarithmic curve-fitting and evaluation of the model at specific domain values outside the given data set.

**Standards Addressed:** AFDA.1; AFDA.2; AFDA.3; AFDA.4

**Data Used:** Data obtained from observation/measurement in activities and imported from Internet databases.

## Materials:

- Applications: **EasyData™** and **Transformation**™, **TI-Interactive™**
- Computer lab
- Graphing calculator and links
- Patty paper
- Graph paper
- Handout – Switch It Up
- Handout – Transforming Functions
- Handout – High/Low Guessing Game
- Handout – Life Expectancy

## Instructional Activities:

### I. Introduction to the unit – Switch It Up

Students will investigate the inverses of functions as an introduction to the logarithmic function as the inverse of the exponential. Students will be able to identify and explain what inverses and logarithms are both algebraically and graphically.

Concepts covered include:

- domain and range;
- continuity;
- linear functions;
- quadratic functions;
- exponential functions;
- logarithmic functions;
- evaluating a function for a value in the domain;
- independent and dependent variables;
- inverses;

- zeros; and
- intercepts.

## II. Transforming Functions

Students will investigate the transformations of quadratic, exponential, and logarithmic functions. Students will identify reflections, translations, and dilations of known parent functions. Students will match equations to graphs using a transformational approach.

Concepts covered include:

- parent functions (linear, quadratic, exponential, logarithmic);
- transformations (translations, reflections, dilations);
- domain and range;
- zeros; and
- intercepts.

## III. High/Low Guessing Game

Students will play the game and develop an optimal strategy which decreases the number of possibilities by half. Students will be able to identify this as an exponential decay function. Students will then investigate a logarithmic function which reverses the process, solving an exponential equation.

Concepts covered include:

- exponential decay;
- logarithmic functions;
- properties of logarithms;
- change of base formula; and
- domain and range.

## IV. Life Expectancy

Students will use the Internet to collect data from the National Center for Health Statistics Web site. Students will analyze data and model various functions.

Concepts covered include:

- scatter plots;
- domain and range;
- continuity;
- function;
- evaluate a function;
- independent and dependent variables;
- linear regression;
- quadratic regression;
- exponential regression;
- logarithmic regression;
- correlation coefficient;
- transformations;
- asymptotes; and
- end behavior.

## Activity I: Teacher Notes--Switch It Up

The logarithmic function is introduced as the inverse of the exponential function. The concept of an inverse function should be reinforced by students' prior knowledge of inverse functions as reflections across $y = x$. The teacher should reinforce previously learned concepts of linear and quadratic functions, domain, and range.

The teacher may choose to incorporate more examples of inverse functions or having students graph more pairs of inverses.

## Switch It Up

### What is the inverse of a function?

The inverse reverses or undoes the previous action. If f(x) represents the original function then $\mathbf{f^{-1}(x)}$ is the inverse function.

Look at the function $\mathbf{f(x) = x + 5}$.
This function adds 5 to a number, so what is the reverse operation or inverse function?
The inverse is to ________________________
$\mathbf{f^{-1}(x)}$ = ____________

Look at the function $\mathbf{g(x) = 2x}$.
This function multiplies a number by 2, so what would be the inverse function?
The inverse is to ________________________
$\mathbf{g^{-1}(x)}$ = ____________

Look at the function $\mathbf{h(x) = x^2}$.
What does this function do? ______________________
The inverse is to _______________________________
$\mathbf{h^{-1}(x)}$ = _____________

1. Fill in the tables for **h(x)** and $\mathbf{h^{-1}(x)}$

| x | h(x) |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |

| x | $h^{-1}(x)$ |
|---|---|
| 0 | |
| 1 | |
| 4 | |
| | 9 |
| | |
| | |
| | |
| | |
| | |

On graph paper, plot the points from the tables and connect the points with a smooth line.

What do you notice about the graphs?

The graphs are mirror images or reflections across the line:  y = _________________

What do you notice about the domain and range of each?

2. Now we will investigate the inverse of an exponential function $\mathbf{f(x) = 3^x}$.
   Fill in the table below. Then sketch a graph of the function.

| x | - 3 | - 2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|
| f(x) | | | | | | | | |

   What is the domain of the exponential function?

   What is the range of the exponential function?

3. Trace the x-axis, y-axis, and the graph of **f(x)** onto a sheet of patty paper. Be sure to label the axes. Reflect the graph of f(x) across the line **y = x** by holding the top-right and bottom left corners of the patty paper in each hand and flipping the sheet of patty paper over. Sketch what you see.

   Is this new graph a function? Explain how you know.

   What is the domain?

   What is the range?

4. Fill in the table below for the new graph.

| x | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| g(x) | - 3 | - 2 | -1 | 0 | 1 | 2 | 3 | 4 |

5. The inverse of an exponential function has a special name called a **logarithmic** function.
   If $\mathbf{b^y = x}$ then we can write $\mathbf{\log_b x = y}$.
   Ex. $\mathbf{5^3 = 125}$ so $\mathbf{\log_5 125 = 3}$
   Fill in the table completely, creating two of your own problems.

| Exponential Form | Logarithmic Form |
|---|---|
| $4^3 = 64$ | $\log_4 64 =$ |
| $5^4 =$ | $\log_5 625 = 4$ |
| $3^4 = 81$ | |
| | $\log_2 128 = 7$ |
| $6^3 =$ | |
| | $\log_2 64 =$ |
| | |
| | |

   On the calculator, you will notice the two logarithmic functions most commonly used, “log” which is base 10 and “ln” which is base *e,* an irrational number.

6. Summarize what you have learned.

How would you explain the inverse of a function to someone else?

How would you explain logarithms to someone else?

## Activity II: Teacher Notes--Transforming Functions

Students will investigate a transformational approach to graphing functions using a graphing technology. The **Transformation™** Application may be used. A TI-interactive could be used in a computer laboratory for a hands-on activity that explores transformational graphing techniques. This activity builds upon students' prior knowledge, reinforces previous concepts, while it develops new understanding of logarithmic functions. Transformation cut-outs can be used as an individual or group activity for enrichment or review.

# Transforming Functions

We have encountered several basic or parent functions such as linear, quadratic, exponential, and logarithmic functions. By using these basic or parent functions, we can build new functions by several types of movements such as dilating, translating, and reflecting the original parent function. We will now investigate these transformations.

For each part, make a sketch of each original function and its transformations on one grid of graph paper. Use different colored pencils to represent each transformation.

A **reflection** is a movement where a graph "flips" over an axis. It is called a **reflection** because it will be a mirror image of the original. We will graph f(x), - f(x), and f(-x)

1. Sketch the graph of each function.
   a. $y = x^2$ b. $y = -x^2$ c. $y = (-x)^2$

2. Sketch the graph of each function.
   a. $y = e^x$ b. $y = -e^x$ c. $y = e^{-x}$

3. Sketch the graph of each function.
   a. $y = \ln x$ b. $y = -\ln x$ c. $y = \ln(-x)$

What do you notice about the graphs of – f(x), in each b above?

What do you notice about the graphs of f(- x), in each c above?

A **dilation** is a transformation that enlarges or shrinks a graph.

4. Sketch the graph of each function.
   a. $y = x^2$ b. $y = 2x^2$ c. $y = 0.5x^2$

5. Sketch the graph of each function.
   a. $y = e^x$ b. $y = 2e^x$ c. $y = 0.5e^x$

6. Sketch the graph of each function.
   a. $y = \ln x$ b. $y = 2 \ln x$ c. $y = 0.5 \ln x$

What do you notice about the graphs of **a f(x)** when a > 1, in each b above?

What do you notice about the graphs of **a f(x)** when a < 1, in each c above?

A **translation** is a transformation that involves sliding a graph vertically or horizontally.

7. Sketch the graph of each function.
   a. $y = x^2$
   b. $y = x^2 + 3$
   c. $y = x^2 - 3$
   d. $y = (x - 3)^2$
   e. $y = (x + 3)^2$

8. Sketch the graph of each function.
   a. $y = e^x$
   b. $y = e^x + 3$
   c. $y = e^x - 3$
   d. $y = e^{x - 3}$
   e. $y = e^{x + 3}$

9. Sketch the graph of each function.
   a. $y = \ln x$
   b. $y = \ln (x) + 3$
   c. $y = \ln (x) - 3$
   d. $y = \ln (x - 3)$
   e. $y = \ln ( x + 3)$

What do you notice about the graphs of $f(x) + k$, in B and C above?

What do you notice about the graphs of $f(x - h)$, in D and E above?

**Combining Transformations**

10. Sketch the graph of $\mathbf{y = (x - 3)^2 + 4}$ and describe the transformations of the parent graph.

11. Sketch the graph of $\mathbf{y = 2e^{x + 4}}$ and describe the transformations of the parent graph.

12. Sketch the graph of $\mathbf{y = -\ln(x - 3) + 1}$ and describe the transformations of the parent graph.

13. Sketch the graph of $\mathbf{y = \ln(-x) - 4}$ and describe the transformations of the parent graph.

14. Sketch the graph of $\mathbf{y = 0.5(x + 4)^2 - 2}$ and describe the transformations of the parent graph.

15. Sketch the graph of $\mathbf{y = -x + 3}$ and describe the transformations of the parent graph.

16. Write the equation of an exponential function that is reflected across the y-axis and translated up 3.

17. Write the equation of a quadratic function that is reflected across the x-axis, translated right 2 and down 4.

18. Write the equation of a logarithmic function that is dilated by a factor of 2 and translated left 1 and up 5.

19. Write the equation of a linear function that is translated to the right 2.

# Transformation Cut-outs

| | | | |
|---|---|---|---|
| Quadratic<br>Reflect x-axis<br>Translate up 2 | Exponential<br>Reflect y-axis<br>Translate up 3 | Logarithmic<br>Translate up 2<br>Translate right 3 | Quadratic<br>Parent function |
| Exponential<br>Parent function | Logarithmic<br>Parent function | Quadratic<br>Dilate by 2<br>Translate left 3 | Exponential<br>Translate down 3<br>Translate left 2 |
| Logarithmic<br>Reflect x-axis<br>Translate down 2 | Quadratic<br>Reflect x-axis<br>Translate down 3<br>Translate left 2 | Exponential<br>Dilate by 3<br>Translate right 2<br>Translate up 4 | Logarithmic<br>Translate left 2<br>Translate down 4 |
| $y = -(x+2)^2 - 3$ | $y = e^{x+2} - 3$ | $y = \ln(x-3) + 2$ | $y = x^2$ |
| $y = e^x$ | $y = -\ln(x) - 2$ | $y = -x^2 + 2$ | $y = e^{-x} + 3$ |
| $y = \ln(x+2) - 4$ | $y = 2(x+3)^2$ | $y = 3e^{x-2} + 4$ | $y = \ln x$ |
| | | | |
| | | | |
| | | | |

**Activity III: Teacher Notes--High/Low Guessing Game**

Students should work with a partner for this activity. Use the activity as a warm-up or introduction to lessons on properties of logarithms, solving exponential equations, or using a change of base formula.

Students will team up with a partner to play the game and determine an optimal strategy, which reduces the possibilities in half each time. This is actually an exponential decay process.

Students will solve an exponential equation by taking the log of both sides. Students may wonder how this formula was derived which leads into a properties of logarithms lesson. The formula is a change of base formula which can be used to find logarithm with any base, $\log_b x = \dfrac{\log x}{\log b}$.

## High/Low Guessing Game

The High/Low Guessing Game requires that one try to determine by means of ***yes*** or ***no*** questions a number chosen between two given numbers.

With a partner, play the game. Start by having one person pick a counting number between one and twenty. The other person must determine strategically what the number is in as few guesses as possible. After the number is guessed, switch roles. How many guesses did it take?

Describe the best strategy to pick the number correctly with the fewest number of guesses.

Now try your strategy by picking a number between one and forty. Work with a partner again.
How many guesses did it take?

How many guesses do you think it would take to pick the correct number if it is between one and one million? Explain your reasoning.

### One and One Million!

Each question can cut the number of possibilities in half. Thus the first question, ***Is it less than or equal to 500,000?*** leaves 1,000,000 (0.5) possibilities. The second question leaves $1{,}000{,}000(0.5)^2$ possibilities and so forth.

To determine number of questions needed to determine a number between 1 and n, we need to solve the equation $\mathbf{2^x = n}$, where x is the number of equations. The left member of the equation is an exponential expression, but the equation can be solved using logarithms. There is no base 2 logarithm key on the calculator, so using properties of logarithms and the change of base formula we can obtain:

$$x = \frac{\log n}{\log 2}$$

To solve for the number of questions to determine a number between 1 and 1,000,000, use the above formula and let n = 1,000,000.
How many questions does it take?

Use the formula to determine how many guesses it would take to determine a number chosen between one and one billion.

## Activity IV: Teacher Notes--Life Expectancy

**Life Expectancy** may be an individual or group activity and requires Internet access. Students will collect data from **National Center for Health Statistics** to find life expectancy at birth depending on year born. Students will discover that a logarithmic function best models the given data overall, but that this model has limitations for extrapolating into the distant past or distant future.

# Life Expectancy

What do you think is the average life expectancy for someone born today and living in the United States all of his or her life?

What do you think was the average life expectancy for someone born in the United States in the year 1900?

What would you predict about the average life expectancy of someone who will be born 20 years from now?

**Collecting Data**

1. Use the Internet to find the **National Center for Health Statistics** Web site. We want to find the **National Vital Statistics Report** and then go to **Final Data 2007** (see link in Resources). We want to examine the data tables for **Life Expectancy**. Collect data for life expectancy at time of birth. Complete the table below.

| Year | Life Expectancy | Year | Life Expectancy |
|---|---|---|---|
| 1940 | | 1996 | |
| 1950 | | 1997 | |
| 1960 | | 1998 | |
| 1970 | | 1999 | |
| 1975 | | 2000 | |
| 1980 | | 2001 | |
| 1985 | | 2002 | |
| 1985 | | 2003 | |
| 1990 | | 2004 | |
| 1995 | | 2007 | |

## Graphing and Determining the Best Model

2. Use the graphing calculator and enter the data above into the **STAT** lists. Display the scatter plot of the data. Use the calculator to determine what equation best models the given data. Round all coefficient values in the equations to the nearest hundredth. For each regression, record the correlation coefficient, ***r***, rounded to nearest ten-thousandth. Make sure that you have **DiagnosticOn**, which can be found by pressing the **2$^{nd}$** and then the **0** key and scrolling down.

Linear Regression Equation:________________________ r = __________

Quadratic Regression Equation:_____________________ $R^2$ = __________

Exponential Regression Equation:___________________ r = __________

Logarithmic Regression Equation:___________________ r = __________

Use **ZoomStat** to set initial viewing screen. Then graph the four models together with the scatter plot. What observations can you make about each model?

What similarities do you notice about the graphs of the functions?

What differences do you notice about the graphs of the functions?

Which model seems to best fit the data?

We want to make sure our model fits the real-world over an extended period of time. Use **ZoomOut** to observe overall behavior and end behavior of the graph.

What do you notice about the quadratic function?

What do you notice about the values of the correlation coefficients and how the graph fits the data?

Which function do you think would best fit the given data?

### Extrapolating Beyond the Given Data

3. Using the logarithmic model, extrapolate, or predict beyond the data, what the average life expectancy would be for each birth year.

| Year | Life Expectancy |
|---|---|
| 2015 | |
| 2020 | |
| 2030 | |
| 3000 | |
| 1800 | |
| 1700 | |
| 1600 | |

What can you say about the model's ability to determine what the life expectancy will be in the future? Do these results make sense?

What can you say about the model's ability to determine what the life expectancy was in the past, before the given data? Do these results make sense?

**Fitting the Equation**

4. Select one of the regression models you performed on the life expectancy data and enter it in to **[Stat] [Edit]** and perform the appropriate regression to determine the curve of best fit. Activate **Stat Plot 1** for the scatter plot (year, life expectancy) graph. Activate **Stat Plot 2** for the scatter plot (year, RESID) graph. Complete the table below.

| **TIME L1** | **ACTUAL DISTANCE L2** | **FITTED DISTANCE L3** | **ACTUAL – FITTED L4 = L2 – L3** | **(ACTUAL – FITTED)$^2$ L5 = (L4)$^2$** |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| Total Actual – Fitted [Sum(L4)] | | | | |
| Total (Actual – Fitted)$^2$ [Sum(L5)] | | | | |

5. Select one of your other choices and determine the regression equation for the given data. Repeat the above process recording the values in the table below.

| **TIME L1** | **ACTUAL DISTANCE L2** | **FITTED DISTANCE L3** | **ACTUAL – FITTED L4 = L2 – L3** | **(ACTUAL – FITTED)$^2$ L5 = (L4)$^2$** |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| Total Actual – Fitted [Sum(L4)] | | | | |
| Total (Actual – Fitted)$^2$ [Sum(L5)] | | | | |

6. Compare and contrast the two tables.

## Other Topics for Exploration in Logarithmic Functions

- Earthquakes and Richter Scale R = log I
- Decibel and Sound Intensity B = 10 Log ($I/I_0$)
- Acidity pH = - log[$H^+$]
- Time it takes to double one's money in an investment
- Linearization of Data and use of logarithmic graph paper
- Logarithmic spirals

## Resources:

The Math Forum – Exploring Data: Exploring Data, Courses and Software
http://mathforum.org/workshops/usi/dataproject/usi.genwebsites.html

- The Data Library – Pat Daley

  This site includes collaborative projects - specific data collection projects that teachers and their students may become a part of; data sets that can be downloaded then sorted, manipulated, and graphed; and other sources of data - sites like the Bureau of Labor Statistics and the Chance Database that offer many more data sets in other formats.

U.S. Census Bureau – International Program
http://www.census.gov/ipc/www/idb/

- International Database

Math Tools – Math Tools is a project of The Math Forum @ Drexel, funded in part by the National Science Foundation.
http://mathforum.org/mathtools/sitemap2/a2/

- Math Topics for Algebra II

Real Time World Statistics - Worldometers
http://www.worldometers.info/

- World statistics updated in real time.

National Center for Health Statistics - The National Center for Health Statistics' website is a rich source of information about America's health.

- National Vital Statistics Report Volume 58, Number 19
  http://www.cdc.gov/NCHS/data/nvsr/nvsr58/nvsr58_19.pdf

Algebra Lab
http://www.algebralab.org

- Algebra Lab is an online learning environment that focuses on topics and skills from high school mathematics.

Math Bits.com - MathBits.com is devoted to offering fun, yet challenging, lessons and activities in secondary (and college level) mathematics and computer programming for students and teachers.
http://mathbits.com/MathBits/TISection/Statistics2/logarithmic.htm

- Finding Your Way around the TI 83+/84+ Graphing Calculator – Logarithmic Regression Model Example - Weeping Higan cherry trees in Washington, D.C.